

# Berry Phase in Neutrino Oscillations

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## Abstract

We study the Berry phase in neutrino oscillations for both Dirac and Majorana neutrinos. In order to have a Berry phase, the neutrino oscillations must occur in a varying medium, the neutrino-background interactions must depend on at least two independent densities, and also there must be CP violation if the neutrino interactions with matter are mediated only by the standard model W and Z boson exchanges which implies that there must be at least three generations of neutrinos. The CP violating Majorana phases do not play a role in generating a Berry phase. We show that a natural way to satisfy the conditions for the generation of a Berry phase is to have sterile neutrinos with active-sterile neutrino mixing, in which case at least two active and one sterile neutrinos are required. If there are additional new CP violating flavor changing interactions, it is also possible to have a non-zero Berry phase with just two generations.

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## I. INTRODUCTION

In the past decade, great progress has been made in the study of neutrinos. One of the most important achievements is the observation of neutrino oscillations[1]. If different species of neutrinos have different masses and also mix with each other neutrino oscillations can occur in vacuum[2] and also in matter[3]. The oscillation effects are closely related to the phases of neutrino fields.

In quantum mechanics, particle wave functions satisfy the Schrödinger equation. Once the Hamiltonian of the system is known, the evolution of a quantum system is determined. Berry has shown that if the Hamiltonian depends on time via a set of adiabatic parameters, besides the usual dynamic phase, a non-dynamic phase (the Berry phase) will also be developed[4]. Neutrino oscillations in matter, where the varying matter density plays the role of the adiabatic parameters, fit such a situation nicely. When the neutrinos move through a medium, a Berry phase is then expected to be generated.

In this work we study the conditions with which a Berry phase can be developed in neutrino oscillations when passing through a medium. We will consider Dirac and Majorana neutrinos with both active and active-sterile neutrino mixing, and also study cases with new CP violating flavor changing interactions. We find that in order to generate a Berry phase ,

- there must be  $CP$  violation,
- but the CP violating phases peculiar to Majorana neutrinos do not contribute,
- there must be three neutrino generations if neutrinos interact with the medium only through W and Z exchanges, and
- the neutrino-background matter interaction must depend on two independent densities, such as the electron and neutron densities (which requires interactions beyond the standard model, or the introduction of sterile neutrinos), or the electron and background neutrino densities (which is possible in such astronomical objects as supernovae).

We find that active and sterile neutrino mixing can provide a natural realization of all the conditions listed above due to different interactions of the Z boson with active and sterile

neutrinos. If there are new interactions of neutrinos with matter, it is possible to have a Berry phase with just two neutrinos.

Several authors have considered the phase properties in neutrino oscillations[5, 6]. Our discussions follow Berry's definition of Berry phase in terms of a parametric dependence of the Hamiltonian, and the phase depends on motion through a loop in parameter space[4]. This is in contrast to Ref.[6] who used a different definition due to Ref.[7] which will not be discussed here.

The conditions of generating a Berry phase with Dirac neutrino oscillation in matter with the standard W and Z interactions have been discussed in Ref.[5]. Our results generalize the discussions in Ref.[5] to include Majorana neutrinos, active and sterile neutrino mixing, and new interactions. Specific examples which can realize all the conditions are discussed.

## II. DIRAC NEUTRINO OSCILLATION AND BERRY PHASE

We start with the study of the Berry phase for Dirac neutrinos going step by step to identify the origin of Berry phase. The discussions in this section also serve to set up notations for the analysis in the other sections. Neutrino oscillations are due to the mismatch of mass and weak interaction eigenstates of the interaction Lagrangian. For Dirac neutrinos the relevant Lagrangian, in the weak eigenstate basis where the charged leptons are already in their mass eigenstate basis, is given by

$$L = \bar{\nu}_L i\gamma^\mu \partial_\mu \nu_L + \bar{\nu}_R i\gamma^\mu \partial_\mu \nu_R - \bar{\nu}_R M \nu_L - \bar{\nu}_L M^\dagger \nu_R - \frac{g}{2\cos\theta_W} \bar{f} \gamma_\mu (g_V^f - g_A^f \gamma_5) f Z^\mu - \frac{g}{\sqrt{2}} \bar{l}_L \gamma^\mu \nu_L W_\mu^- + h.c., \quad (1)$$

where  $g_V = T_3^f - 2\sin^2\theta_W Q_f$  and  $g_A = T_3^f$  with  $f$ ,  $T_3^f$  and  $Q_f$  the Standard Model left-handed fermions and their corresponding isospin and electric charges.

The mis-match of mass and weak eigenstates means that the neutrino mass matrix  $M$  is a non-diagonal matrix.  $M$  can be diagonalized to a diagonal mass matrix  $\hat{M}$  by a bi-unitary transformation, so that  $M = U' \hat{M} U^\dagger$ . Here  $U'$  and  $U$  are unitary matrices. When writing the Lagrangian in the mass eigenstate basis  $\nu_L^m = U^\dagger \nu_L$  and  $\nu_R^m = U'^\dagger \nu_R$ , the form of the first four terms does not change, with  $M$  replaced by  $\hat{M}$ , but the W interaction term in the above equation changes to  $-(g/\sqrt{2}) \bar{l}_L^m \gamma^\mu U \nu_L^m$ .  $U$  is the PMNS[2] mixing matrix, and  $U'$  is the equivalent matrix for the right handed neutrinos which plays no further role in

our analysis as it does not enter the standard model interactions. By redefining the lepton field phases, for  $n$  generations of leptons, the mixing matrix  $U$  can be parameterized by  $n(n-1)/2$  rotation angles and  $(n-1)(n-2)/2$  CP violating phases.

A weak eigenstate of neutrino  $\nu_i$  of energy  $E$  produced in association with a charged lepton would propagate in vacuum as

$$\nu_i(t) = \sum_j U_{ij} e^{-ix \cdot p_j} \nu_j^m(0) = \sum_{jk} U_{ij} U_{kj}^* \nu_k(0) e^{-ix \cdot p_j}. \quad (2)$$

Here  $p_j = (E, \vec{p}_j)$  and  $|\vec{p}| = \sqrt{E^2 - m_j^2} \approx E - m_j^2/2E$ .

The probability amplitude for observing a  $\nu_k$  neutrino at time  $t$  after the creation of a  $\nu_i$  neutrino, a distance  $L$  away in the direction of propagation, is given by

$$A(\nu_i \rightarrow \nu_k) = \sum U_{ij} U_{kj}^* e^{-ix \cdot p_j} \approx e^{-iE(t-L)} \sum U_{ij} U_{kj}^* e^{-im_j^2 L/2E}. \quad (3)$$

$|A(\nu_i \rightarrow \nu_k)|^2$  is the transition probability for  $\nu_i$  to  $\nu_k$ .

The phases  $-x \cdot p_j$  in the above are usually referred as dynamic phases. Since we follow Berry's definition of Berry phase in terms of a parametric dependence of the Hamiltonian, and the phase depends on motion through a loop in parameter space[4], the dynamic phase  $-x \cdot p$  will not contribute to the Berry phase. In order to study the Berry phase defined in Ref.[4] one must separate the effects of the dynamic phases and identify phases which depend on some slowly varying adiabatic parameters in a given system. In the case of neutrino oscillation, we find that the slowly varying matter background can be identified as the adiabatic parameter.

In medium with a finite matter density neutrino propagation is different to that in vacuum because of the interaction of neutrinos with background matter. One can obtain the equation of motion in this case by integrating out the  $W$  and  $Z$  in eq. (1) obtaining the four fermion interaction term  $-\sqrt{2}G_F \bar{\nu}_L \gamma^\mu (j_\mu N + j'_\mu N') \nu_L$  in the Lagrangian. In the Standard Model  $N$  and  $N'$  are diagonal matrices representing interaction of neutrinos with the medium due to  $W$  and  $Z$  exchange respectively. For the purpose of discussing neutrino oscillations in the sun and in the earth which are unpolarized,  $j_\mu = \bar{e} \gamma_\mu e$  is the electron number density current and  $N = \text{diag}(1, 0, 0)$ . This is due to  $W$  exchange between the neutrinos and the background electrons.  $N'$  is a unit matrix with  $j'_\mu = \sum_f \bar{f} \gamma_\mu f (T_3^f - 2Q_f \sin^2 \theta_W)$  generated by

Z exchange. The term proportional to  $N'$  does not affect the mixing of the active neutrinos, and will therefore be ignored until section V. A non-trivial effect will show up there when we discuss oscillations with sterile neutrinos.

The equations of motion are given by

$$\begin{aligned} i\gamma^\mu \partial_\mu \nu_L - M^\dagger \nu_R - \sqrt{2}G_F j_\mu \gamma^\mu N \nu_L &= 0, \\ i\gamma^\mu \partial_\mu \nu_R - M \nu_L &= 0. \end{aligned} \quad (4)$$

For a static case,  $j_\mu = \rho(x)g_{\mu 0}$ . Writing  $\nu_L = e^{-iEt}\psi(x)$  for fixed energy  $E$  much larger than  $m_i$ , and assuming slowly varying density  $(\rho E)^{-1}d\rho/dx \ll 1$ , the equation of motion for  $\psi$  for neutrino beam travelling along x direction is given by[8]

$$[E^2 - M^\dagger M + \partial_x \partial_x - 2\sqrt{2}G_F \rho EN]\psi(x) = 0. \quad (5)$$

We have used the approximation  $\gamma^0 \gamma^i (-i\partial_i)\psi(x) \approx -E\psi(x)$  for  $E \gg m_i$ .

The above equation can be written as  $(id/dx + (E^2 - M^\dagger M - 2\sqrt{2}G_F \rho EN)^{1/2})(id/dx - (E^2 - M^\dagger M - 2\sqrt{2}G_F \rho EN)^{1/2})\psi(x) = 0$ , where the assumption of slow variation is again used. For not too large density  $\rho(x)$ ,  $(E^2 - M^\dagger M - 2\sqrt{2}G_F \rho EN)^{1/2} \approx E - M^\dagger M/2E - \sqrt{2}G_F \rho N$ , and one obtains the usual equation of motion for neutrinos in matter[3],

$$i\frac{d}{dx}\psi(x) = -H\psi(x), \quad H = E - \frac{M^\dagger M}{2E} - AN, \quad (6)$$

where  $A = \sqrt{2}G_F \rho$ .

One can write  $H$  in the following form

$$\begin{aligned} H &= \tilde{E} - \tilde{A}, \\ \tilde{E} &= E - \frac{1}{2E} \frac{m_1^2 + m_2^2 + m_3^2}{3} - \frac{A}{3}, \quad \tilde{A} = \frac{1}{2E} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12}^* & a_{22} & a_{23} \\ a_{13}^* & a_{23}^* & a_{33} \end{pmatrix} \\ a_{ij} &= \frac{1}{3}(\Delta m_{21}^2 - \Delta m_{32}^2)\delta_{ij} - \Delta m_{21}^2 U_{i1} U_{j1}^* + \Delta m_{32}^2 U_{i3} U_{j3}^* \\ &+ \frac{1}{3}2EA(2\delta_{i1}\delta_{j1} - \delta_{i2}\delta_{j2} - \delta_{i3}\delta_{j3}). \end{aligned} \quad (7)$$

Note that  $\tilde{A}$  is a traceless Hermitian matrix.

Writing  $\psi = e^{i\int_0^x \tilde{E} dx} \tilde{\psi}$ , we find that  $\tilde{\psi}$  satisfies the following equation of motion

$$i\frac{d}{dx}\tilde{\psi} = \tilde{A}\tilde{\psi}. \quad (8)$$

For a uniform medium one obtains a solution for  $\nu(t)$  similar to eq.(2), but with  $U$  replaced by the unitary matrix  $\tilde{U}$  which diagonalizes  $\tilde{A}$ , that is,  $\tilde{A} = \tilde{U}\hat{A}\tilde{U}^\dagger$ , and  $E - m_i^2/2E$  by  $\tilde{E} - \lambda_i/2E$  with  $\lambda_i/2E$  being the eigenvalues of  $\tilde{A}$ . The eigenvalues are given by

$$\begin{aligned} \lambda_1 &= 2s \cos(\theta/3), \quad \lambda_2 = 2s \cos(\theta/3 + 2\pi/3), \quad \lambda_3 = 2s \cos(\theta/3 - 2\pi/3), \quad \theta = \arccos(t^3/s^3), \\ 2t^3 &= \text{Det}(2E\tilde{A}), \quad 3s^2 = |a_{12}|^2 + |a_{13}|^2 + |a_{23}|^2 + a_{11}^2 + a_{22}^2 + a_{11}a_{22}. \end{aligned} \quad (9)$$

In this case all phases are dynamic again. No Berry phases are generated.

In a non-uniform medium, a Berry phase may be generated. In this case the matrix  $\tilde{U}$  is now  $x$  dependent through its dependence on  $\rho(x)$ . Writing the wave function  $\tilde{\psi}(x)$  in the diagonal basis of  $\tilde{A}$ , we have

$$i\frac{d}{dx}\psi^m = (\hat{A} - i\tilde{U}^\dagger \frac{d}{dx}\tilde{U})\psi^m, \quad (10)$$

where  $\psi^m = \tilde{U}^\dagger \tilde{\psi}$ .

Taking the second term on the right in the above equation as perturbation, in the adiabatic approximation, one obtains

$$\nu(t, x)_i = e^{-iEt} \sum_j \tilde{U}_{ij} e^{+i\gamma_j} e^{i\int_0^x (\tilde{E} - \lambda_j/2E) dx} \nu_j^m(0), \quad (11)$$

where  $\gamma_j$  is given by

$$\gamma_j = i \int_0^L (\tilde{U}^\dagger \frac{d}{dx} \tilde{U})_{jj} dx = i \int_C (\tilde{U}^\dagger \vec{\nabla} \tilde{U})_{jj} d\rho. \quad (12)$$

In the above we have allowed (as will be important later) for the density parameter to be multi-dimensional. “ $\vec{\nabla}$ ” is the gradient taken in density parameter space  $\vec{\rho}$ . The position  $L$  is chosen so that  $\vec{\rho}(0) = \vec{\rho}(L)$ , following Berry’s prescription, and  $C$  is a closed curve in parameter space. This is the Berry phase in neutrino oscillation.

### III. CONDITIONS FOR A NON-ZERO BERRY PHASE

Using the above definition of Berry phase in neutrino oscillation, certain conditions have to be satisfied to generate a non-zero Berry phase. We now discuss these conditions.

The matrix  $\tilde{U}$  must contain complex phases in order to have a non-zero  $\gamma_j$ . Therefore there must be at least three neutrino mixing to have a Berry phase since with two neutrino mixing with just W and Z interactions, the matrix  $U$  can always be made real.

A non-zero  $(\tilde{U}^\dagger \nabla_\rho \tilde{U})_{jj}$  is not a sufficient condition for a non-zero  $\gamma_j$ . It is important to realize that the matrix  $\tilde{U}$  is not uniquely defined. Because the matrix  $\hat{A}$  is diagonal, if  $\tilde{P} = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3})$ , then

$$\tilde{P}\hat{A}\tilde{P}^* = \hat{A}, \quad \text{thus} \quad \tilde{A} = \tilde{U}\hat{A}\tilde{U}^\dagger = \tilde{U}\tilde{P}\hat{A}\tilde{P}^*\tilde{U}^\dagger. \quad (13)$$

Thus both  $\tilde{U}$  and  $\tilde{U}' = \tilde{U}\tilde{P}$  diagonalize the matrix  $\tilde{A}$ , but as

$$(\tilde{U}'^\dagger \vec{\nabla} \tilde{U}')_{jj} = (\tilde{U}^\dagger \vec{\nabla} \tilde{U})_{jj} + i\vec{\nabla}\theta_j, \quad (14)$$

after integrating over the closed loop  $C$ , the term proportional to  $\vec{\nabla}\theta$  vanishes. Thus the Berry phase is independent of the choice of the matrix  $\tilde{U}$ . One may regard the transformation  $\tilde{U} \rightarrow \tilde{U}\tilde{P}$  as a gauge transformation, and the Berry phase is a gauge independent observable.

The phase  $\theta_j$  can be viewed as a pure gauge transformation of Berry phase. If one can find a gauge in which the Berry phase is zero before integrating over  $C$ , the phase is not physical and can be gauged away.

We illustrate this phenomenon by considering the case of three generations. One way to obtain the matrix  $\tilde{U}$  is by requiring

$$(2E\tilde{A} - \lambda_i) \begin{pmatrix} \tilde{U}_{1i} \\ \tilde{U}_{2i} \\ \tilde{U}_{3i} \end{pmatrix} = 0. \quad (15)$$

For the  $AN$  given in eq.(7), using the first two rows in eq.(15), we can write

$$\begin{pmatrix} \tilde{U}_{1i} \\ \tilde{U}_{2i} \\ \tilde{U}_{3i} \end{pmatrix} = \frac{1}{N_i} \begin{pmatrix} (a_{22} - \lambda_i)a_{13} - a_{23}a_{12} \\ (a_{11} - \lambda_i)a_{23} - a_{13}a_{12}^* \\ a_{12}a_{12}^* - (a_{22} - \lambda_i)(a_{11} - \lambda_i) \end{pmatrix}. \quad (16)$$

Here  $N_i$  is the normalization constant. One obtains

$$\begin{aligned} (\tilde{U} \frac{d}{dx} \tilde{U}^\dagger)_{ii} &= -i \frac{1}{N_i^2} \text{Im}(a_{13}a_{12}^*a_{23}^*) \frac{d}{dx}(a_{22} - a_{11}) \\ &= i\Delta m_{21}^2 \Delta m_{31}^2 \Delta m_{32}^2 \text{Im}(U_{21}U_{23}^*U_{33}U_{31}^*) \frac{1}{N_i^2} \frac{d}{dx}(a_{22} - a_{11}). \end{aligned} \quad (17)$$

Since  $d(a_{22} - a_{11})/dx = -(8\sqrt{2}/3)EG_F d\rho/dx \neq 0$ , it seems that a non-zero Berry phase has been generated.

On the other hand, one can also obtain  $\tilde{U}_{ij}$  by using the last two rows of eq. (15) to obtain

$$\begin{pmatrix} \tilde{U}'_{1i} \\ \tilde{U}'_{2i} \\ \tilde{U}'_{3i} \end{pmatrix} = \frac{1}{\tilde{N}_i} \begin{pmatrix} a_{23}a_{23}^* - (a_{22} - \lambda_i)(a_{33} - \lambda_i) \\ (a_{33} - \lambda_i)a_{12}^* - a_{23}a_{13}^* \\ (a_{22} - \lambda_i)a_{13}^* - a_{12}^*a_{23}^* \end{pmatrix}. \quad (18)$$

One then obtains

$$\begin{aligned} (\tilde{U}' \frac{d}{dx} \tilde{U}'^\dagger)_{ii} &= -i \frac{1}{\tilde{N}_i^2} \text{Im}(a_{13}a_{12}^*a_{23}^*) \frac{d}{dx}(a_{33} - a_{22}) \\ &= i \Delta m_{21}^2 \Delta m_{31}^2 \Delta m_{32}^2 \text{Im}(U_{21}U_{23}^*U_{33}U_{31}^*) \frac{1}{\tilde{N}_i^2} \frac{d}{dx}(a_{33} - a_{22}). \end{aligned} \quad (19)$$

For the  $AN$  given before  $d(a_{33} - a_{22})/dx = 0$ , this gives a zero Berry phase.

However, when we integrate the non-zero result in eq. (17) from  $x = 0$  to  $x = L$  the result is proportional to  $\rho(L) - \rho(0)$  which vanishes. No Berry phase can be generated with one varying density.

In order to generate a non-zero Berry phase, one must go to a multi-dimensional parameter space. This can be achieved if  $a_{ii}$  depends on more than one density  $\rho_i$ . This may happen if neutrinos interact with quarks through beyond the SM interactions, or if there are sterile neutrinos mixed with active neutrinos, and therefore the oscillation in medium depends on the neutron density also. In some astronomical and cosmological circumstances, there may be significant electron neutrino densities [12], as well as a background electron densities. Although the neutrino-neutrino interactions are generated by  $Z$  exchange, when electron neutrinos interact with electron neutrinos both direct and exchange diagrams contribute, while when neutrinos of other flavors interact with the electron neutrino background, only the direct diagram is possible. Thus the neutrino-background neutrino interaction is not proportional to a unit matrix in  $N$ . In the early universe, significant muon and tauon densities may occur. For all of these cases that the interaction matrix depends on more than one density. The Berry phase developed can then be written as,

$$\gamma_j = i \int_{\vec{\rho}(0)}^{\vec{\rho}(L)} (\tilde{U}^\dagger \vec{\nabla} \tilde{U})_{jj} \cdot d\vec{\rho}. \quad (20)$$

After a cyclic motion, one obtains

$$\gamma_j = i \oint (\tilde{U}^\dagger \vec{\nabla} \tilde{U})_{jj} \cdot d\vec{\rho} = i \int_S \vec{\nabla} \times (\tilde{U}^\dagger \vec{\nabla} \tilde{U})_{jj} \cdot d\vec{S},$$



$$= i \int_S (\vec{\nabla} \tilde{U}^\dagger \times \vec{\nabla} \tilde{U})_{jj} \cdot d\vec{S}, \quad (21)$$

where  $\vec{S}$  indicates the area enclosed by the path  $\vec{\rho}$  after a cyclic motion. Note that any pure gauge terms are eliminated since a curl is taken, and  $\vec{\nabla} \times \vec{\nabla} \theta_j = 0$ . Alternatively, we may say that a non-zero value of  $\gamma_j$  indicates that the function  $(\tilde{U}^\dagger \vec{\nabla} \tilde{U})_{jj}$  is not a perfect derivative. We find the Berry phase after a cyclic motion to be given by, for both of our examples in eqs. (16) and (18),

$$\gamma_i = \frac{2}{3} \Delta m_{21}^2 \Delta m_{31}^2 \Delta m_{32}^2 \text{Im}(U_{21} U_{23}^* U_{31}^* U_{33}) \int_S \frac{\lambda_i (\vec{\nabla} a_{33} \times \vec{\nabla} a_{22}) \cdot d\vec{S}}{(\lambda_i^2 - s^2)^3}. \quad (22)$$

The above result agrees with that obtained in Ref.[5].

Assuming the interaction matrix  $AN$  depends on two varying independent densities,  $\rho$  and  $\rho'$ , with the entries  $\sqrt{2}G_F(\alpha_{ii}\rho + \alpha'_{ii}\rho')$  (traceless), we have

$$(\vec{\nabla} a_{33} \times \vec{\nabla} a_{22}) \cdot d\vec{S} = (2\sqrt{2}EG_F)^2(\alpha_{11}\alpha'_{22} - \alpha_{22}\alpha'_{11})dS. \quad (23)$$

It is clear, from the above discussions, that in order to have non-zero values for the Berry phase  $\gamma_i$  in neutrino oscillation, there must be at least three generations of neutrinos with non-vanishing CP violating phases, and the neutrino must propagate through a background medium with which it interacts through at least two independent densities.

#### IV. MAJORANA NEUTRINO OSCILLATION

We have seen in the previous discussions that CP violating phase in the mixing matrix plays an important role in generating a non-zero Berry phase. In the case of Dirac neutrino this requires that there are at least three generations of neutrinos. In the case of Majorana neutrino, there are CP violating phases even with two generations. One might naively expect a non-zero Berry phase with two generations. We now clarify whether CP violating Majorana phases can play a role in generating a non-zero Berry phase. The relevant Lagrangian for Majorana neutrinos is given by

$$\begin{aligned} L = & \bar{\nu}_L i \gamma^\mu \partial_\mu \nu_L - \frac{1}{2} \bar{\nu}_L^c M \nu_L - \frac{1}{2} \bar{\nu}_L M^\dagger \nu^c \\ & - \frac{g}{2 \cos \theta_W} \bar{f} \gamma_\mu (g_V^f - g_A^f \gamma_5) f Z^\mu - \frac{g}{\sqrt{2}} \bar{l}_L \gamma^\mu \nu_L W_\mu^- + h.c., \end{aligned} \quad (24)$$

where  $\nu_L^c = C\bar{\nu}_L^T$  and  $C = i\gamma^2\gamma^0$ .

The mass matrix  $M$  is symmetric in this case, and can be diagonalized in the following way  $M = \sigma^* U^* P^* \hat{M} P^* U^\dagger \sigma^*$ , where  $\sigma = \text{diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3})$  and  $P = \text{diag}(1, e^{i\alpha_2}, e^{i\alpha_3})$  are diagonal phase matrices. The phases in  $\sigma$  can be absorbed into redefinition of charged lepton fields. In the neutrino mass eigenstate basis, the charged current interaction term can be written as  $-(g/\sqrt{2})\bar{l}_L\gamma^\mu U P \nu_L W_\mu^-$ .

When neutrinos pass through a medium, an interacting term  $-\sqrt{2}G_F j_\mu \bar{\nu}_L \gamma^\mu N \nu_L$  needs to be added to the Lagrangian. One obtains the equations of motion as,

$$\begin{aligned} i\gamma^\mu \partial_\mu \nu_L - M^\dagger \nu_L^c - \sqrt{2}G_F j_\mu \gamma^\mu N \nu_L &= 0, \\ i\gamma^\mu \partial_\mu \nu_L^c - M \nu_L + \sqrt{2}G_F j_\mu \gamma^\mu N \nu_L^c &= 0. \end{aligned} \quad (25)$$

Expressing the above equations in the form involving just  $\nu_L$ , we have

$$\begin{aligned} &[-\partial^2 - M^\dagger M - \sqrt{2}G_F i\gamma^\mu \partial_\mu (j_\nu \gamma^\nu N) \\ &+ \sqrt{2}G_F j_\nu \gamma^\nu M^\dagger N (M^\dagger)^{-1} (i\gamma^\mu \partial_\mu - \sqrt{2}G_F j_\mu \gamma^\mu N)]\nu_L = 0. \end{aligned} \quad (26)$$

In the above we have assumed that none of the neutrinos has zero mass. We have in mind to see if there are just two generations of neutrinos a non-zero Berry phase can be generated. With two generations, if one of the neutrinos has zero mass, the Majorana phases in  $P$  can be completely removed and therefore no Berry phase can be developed. We need to discuss the case where  $M^{-1}$  exists.

The first three terms in eq. (26) are the same as the equation of motion for Dirac neutrinos. Since  $M^\dagger M = U P \hat{M}^* P U^T * U^* P^* \hat{M} P^* U^\dagger = U \hat{M}^2 U^\dagger$ , the Majorana phases do not appear in the first three terms. The Majorana phases may appear in the additional two terms. However, we note that the term  $(i\gamma^\mu \partial_\mu - \sqrt{2}G_F j_\mu \gamma^\mu N)\nu_L$  is of order  $M^\dagger \nu_L^c$ . Compared with the third term there is a suppression factor  $M/E$ . For practical applications,  $M/E$  is much smaller than one and can be safely neglected. With this approximation, one therefore concludes that no effect of Majorana phases will show up in neutrino oscillations. One obtains the same equation of motion for Majorana neutrinos as for Dirac neutrinos with the same approximation: CP violating Majorana phases do not play a role in neutrino oscillation[9, 10]. We do not agree with the equation of motion for Majorana neutrinos obtained in Ref.[11].

## V. ACTIVE-STERILE NEUTRINO OSCILLATION

The above discussions clearly show that in order to have a non-zero Berry phase, there must be at least three generations of neutrinos no matter whether they are Dirac or Majorana neutrinos. Can the situation be changed with further modifications? In the following we consider two examples where a non-zero Berry phase can be developed with just two active neutrinos.

Our first example involves active and sterile neutrino oscillations. Light sterile neutrinos  $\nu_R^i$  may be needed if the LSND result[13] for neutrino oscillation is confirmed. The Lagrangian describing light left-handed active and light sterile neutrinos is similar to eq. (24), but with  $\nu_L$  replaced by  $(\nu_L, \nu_R^c)^T$  and the mass matrix  $M$  replaced by

$$M = \begin{pmatrix} M_{LL} & M_D^T \\ M_D & M_{RR} \end{pmatrix} \quad (27)$$

where the different terms are defined by terms in the Lagrangian:  $-(1/2)\bar{\nu}_L^c M_{LL} \nu_L$ ,  $-\bar{\nu}_R M_D \nu_L$  and  $-(1/2)\nu_R M_R \nu_R^c$ .

The matrices  $N$  and  $N'$  are still diagonal, with  $N = \text{diag}(1, 0, 0, \dots, 0)$ , and  $N' = \text{diag}(I_{n_a}, O_{n_s})$ . Here  $n_a$  and  $n_s$  are the numbers of the active and sterile neutrinos. The  $n \times n$  matrices  $I_n$  and  $O_n$  are a unit matrix and a matrix with all elements equal to zero, respectively. The matrix  $N$  plays the same role as discussed earlier. The matrix  $N'$  which was ignored can no longer be ignored because it is not a unit matrix and will affect mixing in matter[10]. To have some specific idea on how  $N'$  affects oscillation, let us consider a simple case with two active and one sterile neutrino oscillation. This is effectively a three generation oscillation with the  $N'$  term included.

One can obtain the equations of motion for the present case by replacing several quantities in relevant equations. These are

1. replacing  $AN$  in eq. (6) by  $AN + BN'$ , where  $B = \sqrt{2}G_F\rho'$  with  $\rho' = \sum_f \rho_f(T_3^f - 2\sin^2\theta_W Q_f) = (-1/2 + 2\sin^2\theta_W)\rho + (1/2 - 2\sin^2\theta_W)\rho_p + (-1/2)\rho_n$  with  $\rho$ ,  $\rho_p$  and  $\rho_n$  being the background electron, proton and neutron number densities, respectively;
2. replacing  $A/3$  in eq. (7) by  $A/3 + n_a B/3$  in the expression for  $\tilde{E}$ ;

3. replacing the term in  $2E\tilde{A}$  proportional to  $A$  by  $2\sqrt{2}EG_F(AN_{ij} + BN'_{ij} - (1/3)(A + n_a B)\delta_{ij})$ .

In particular,

$$a_{ij} = \frac{1}{3}(\Delta m_{21}^2 - \Delta m_{32}^2)\delta_{ij} - \Delta m_{21}^2 U_{i1}U_{j1}^* + \Delta m_{32}^2 U_{i3}U_{j3}^* + \frac{1}{3}2E\sqrt{2}G_F(2\delta_{i1}\delta_{j1}(\rho + \frac{1}{2}\rho') - \delta_{i2}\delta_{j2}(\rho - \rho') - \delta_{i3}\delta_{j3}(1 + 2\rho')). \quad (28)$$

An interesting feature in the present case is that in the diagonal entries  $a_{ii}$  of the Hamiltonian, more than one densities naturally appear which is a necessary condition for Berry phase. Assuming two active and one sterile neutrino mixing with the neutron density to be independent from the electron density in a neutral medium, like the sun and the earth, we obtain

$$(\vec{\nabla}a_{33} \times \vec{\nabla}a_{22}) \cdot d\vec{S} = (2\sqrt{2}EG_F)^2 \frac{1}{6}dS. \quad (29)$$

Here the surface is the one spanned by the densities  $(\rho, \rho_n)$  for the closed loop “C”.

It is possible to have a non-zero Berry phase in oscillations of two active and one sterile neutrinos, as long as there are CP violating phases in the mixing matrix. One may wonder if there is a Berry phase with one active and two sterile neutrino mixing. The result is negative. In both cases, two active and one sterile, and one active and two sterile neutrino mixing cases, the mixing matrix  $\tilde{U}$  contains CP violating phases. However the later case has  $a_{22} = a_{33}$  which results in a zero Berry phase as can be seen from eq. (19). Obviously with more numbers of active and sterile neutrinos mixing, the conditions for generating a non-zero Berry phase can be realized.

## VI. NEW INTERACTIONS

We finally consider a case with new interactions. These interactions may be CP violating and flavor changing. A possible form of interaction interesting to us is R-parity violating interactions[14],  $\frac{1}{2}\lambda_{ijk}L_L^i L_L^j E_R^{C,k}$ ,  $\lambda'_{ijk}L_L^i Q_L^j D_R^{C,k}$ , where  $L_L$ ,  $E_R$ ,  $Q_L$  and  $D_R^c$  are the chiral lepton doublet, charged lepton singlet, quark doublet and down quark singlet in the supersymmetric extension of SM. The quark super-multiplets are also color triplet. i, j, and k are

flavor indices.  $\lambda_{ijk}$  is anti-symmetric in exchanging the first two indices. This interaction will not change the PMNS mixing matrix and neutrino masses, but will change the neutrino interaction in matter. For example, exchange of right-handed sleptons and squarks can generate an interaction Lagrangian given by

$$L_{int} = \frac{\lambda_{1ik}\lambda_{1jk}^*}{2m_{\tilde{e}_R^k}^2} \bar{\nu}_L^j \gamma^\mu \nu_L^i \bar{e}_L \gamma_\mu e + \frac{\lambda'_{i1k}\lambda_{j1k}^*}{2m_{\tilde{d}_R^k}^2} \bar{\nu}_L^j \gamma^\mu \nu_L^i \bar{d}_L \gamma_\mu d_L. \quad (30)$$

The first term in the above Lagrangian has the same form as exchange of W between electron and neutrino. The second term is different. When neutrinos are passing through matter, this term will generate a term proportional to the neutron and/or proton density in the Hamiltonian. The neutron density can be independent from the electron density. For two generation case, the interaction matrix  $AN$  defined earlier is modified to be

$$AN = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} = \sqrt{2}G_F \begin{pmatrix} \rho\alpha_{11} + \rho'\alpha'_{11} & \rho\alpha_{12}e^{i\delta_{12}} + \rho'\alpha'_{12}e^{i\delta'_{12}} \\ \rho\alpha_{12}e^{-i\delta_{12}} + \rho'\alpha'_{12}e^{-i\delta'_{12}} & \rho\alpha_{22} + \rho'\alpha'_{22} \end{pmatrix}, \quad (31)$$

where  $\alpha_{ij} = |\lambda_{1ik}\lambda_{1jk}^*|/4\sqrt{2}G_F m_{\tilde{e}_R^k}^2$ , and  $\delta_{12}$  is the phase of  $\lambda_{1ik}\lambda_{1jk}^*/m_{\tilde{e}_R^k}^2$ . Similarly for the primed quantities. CP is violated if  $\sin\delta_{12} \neq 0$  or  $\sin\delta'_{12} \neq 0$ .  $\rho$  is the electron density in the matter.  $\rho'$  includes proton and neutron densities in matter which come from the second term in the above Lagrangian.  $\rho'$  can be independent from  $\rho$ . There are constraints on the allowed size for the R-parity violating interactions[14].  $\alpha_{ij}$  and  $\alpha'_{ij}$  can be as large as a percent. Since our purpose is to demonstrate the possibility of having a non-zero Berry phase with just two neutrino generation, the actual number is not important for us.

The quantities  $\tilde{E}$  and  $\tilde{A}$  in eq.(7) for the above case are given by

$$\begin{aligned} \tilde{E} &= E - \frac{1}{2}(m_1^2 + m_2^2) - \frac{1}{2}(A_{11} + A_{22}), \quad \tilde{A} = \frac{1}{2E} \begin{pmatrix} a_{11} & a_{12} \\ a_{12}^* & -a_{11} \end{pmatrix}, \\ a_{11} &= -\frac{1}{2}\Delta m_{21}^2 \cos(2\theta) - \frac{1}{2}2E(A_{22} - A_{11}), \\ a_{12} &= \frac{1}{2}\Delta m_{21}^2 \sin(2\theta) + 2EA_{12}. \end{aligned} \quad (32)$$

where  $\theta$  is the vacuum mixing angle. The eigenvalues of the matrix  $2E\tilde{A}$  are  $\lambda_1 = \sqrt{a_{11}^2 + |a_{12}|^2}$  and  $\lambda_2 = -\sqrt{a_{11}^2 + |a_{12}|^2}$ .

We find

$$\begin{aligned}
\gamma_j &= i \oint (\tilde{U}^\dagger \vec{\nabla} \tilde{U})_{jj} \cdot d\vec{\rho} = i \oint \frac{a_{12}^* \vec{\nabla} a_{12} - a_{12} \vec{\nabla} a_{12}^*}{4\lambda_j(\lambda_j - a_{11})} \cdot d\vec{\rho} \\
&= -i \int_S \frac{1}{4\lambda_j^3} [a_{11} \vec{\nabla} a_{12}^* \times \vec{\nabla} a_{12} + \vec{\nabla} a_{11} \times (a_{12} \vec{\nabla} a_{12}^* - a_{12}^* \vec{\nabla} a_{12})] \cdot d\vec{S} \\
&+ \frac{i}{4} \int_S \frac{|a_{12}|^2}{\lambda_j(\lambda_j - a_{11})} \vec{\nabla} \times \frac{a_{12}^* \vec{\nabla} a_{12} - a_{12} \vec{\nabla} a_{12}^*}{|a_{12}|^2} \cdot d\vec{S}.
\end{aligned} \tag{33}$$

To demonstrate that indeed the above can produce a non-trivial geometric phase, let us consider a simple case where  $a_{11} = a$  and  $a_{12} = be^{i\omega x}$  with  $a$  and  $b$  constants. Integrating over one period,  $x = 2\pi/\omega$ , the geometric phase is given by

$$\gamma_1 = -\pi \left(1 + \frac{a}{\sqrt{a^2 + b^2}}\right), \quad \gamma_2 = -\pi \left(1 - \frac{a}{\sqrt{a^2 + b^2}}\right). \tag{34}$$

Note that in the above example CP is violated because  $a_{12}$  is complex.

We see that a non-zero geometric Berry phase can be developed in two neutrino oscillation if there are at least two independent varying matter densities, and also a non-zero CP violating phase difference in the off diagonal elements of the interaction matrix  $AN$ . The case can be easily generalized to the case with three neutrinos.

## VII. CONCLUSIONS

We have studied Berry phase in neutrino oscillations for both Dirac and Majorana neutrinos. In order to have a non-zero Berry phase there must exist at least three generations of neutrinos with CP violation in the mixing matrix and the oscillation must occur in a background with more than one varying densities if the interaction with matter is due to the standard model W and Z exchange. CP violating Majorana phases do not play a role in generating a Berry phase.

If neutrino oscillations involve only active neutrinos, the interaction of Z boson exchange does not affect neutrino oscillations in matter. If there is active and sterile neutrino mixing, the situation changes and Z boson exchange does affect neutrino oscillations in matter. This scenario provides a natural setting to realize the condition of more than one density in matter since the Z boson exchange is sensitive to electron and also neutron densities in matter.

With new CP violating and flavor changing interactions, it is possible to have non-zero Berry phases even with just two generations, but the interactions must still depend on two independent densities in the background medium.

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